Multiple Regression: Estimation and Inference

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The Model

- Multiple regression:

$$\underset{N\times 1}{\mathbf{Y}} = \underset{N\times K}{\mathbf{X}}\underset{K\times 1}{\beta} + \underset{N\times 1}{\mathbf{u}}$$

- or:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + ... + \beta_K X_{Ki} + u_i$$

- or:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} 1 & X_{11} & X_{21} & \cdots & X_{K1} \\ 1 & X_{12} & X_{22} & \cdots & X_{K2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{1N} & X_{2N} & \cdots & X_{KN} \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}$$

Estimating β

- Residuals:

$$\mathbf{u} = \mathbf{Y} - \mathbf{X}\beta$$

- The inner product of $\mathbf{u}$:

$$\begin{aligned}
\mathbf{u}\mathbf{u}' &= \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \\
&= u_1^2 + u_2^2 + ... + u_N^2 \\
&= \sum_{i=1}^{N} u_i^2
\end{aligned}$$

Estimating β

- We want to minimize the squared erros, so start with:

$$\begin{aligned}\mathbf{u}'\mathbf{u} &= (\mathbf{Y} - \mathbf{X}\beta)'(\mathbf{Y} - \mathbf{X}\beta) \\ &= \mathbf{Y}'\mathbf{Y} - 2\beta'\mathbf{X}'\mathbf{Y}' + \beta'\mathbf{X}'\mathbf{X}\beta\end{aligned}$$

- Now get:

$$\frac{\partial \mathbf{u}'\mathbf{u}}{\partial \beta} = -2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\beta$$

Estimating β

- Solve:

$$
\begin{aligned}
-2\mathbf{X}'\mathbf{Y} + 2\mathbf{X}'\mathbf{X}\beta &= 0 \\
-\mathbf{X}'\mathbf{Y} + \mathbf{X}'\mathbf{X}\beta &= 0 \\
\mathbf{X}'\mathbf{X}\beta &= \mathbf{X}'\mathbf{Y} \\
(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\
\beta &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}
\end{aligned}
$$

- **Important Note: Unlike bivariate OLS, we do not compute the estimates using $(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$**

Assumptions of the CLRM

1. Linearity
   - The CLRM as specified in the form $Y_i = \beta_1 + \beta_2 X_{2i} + \ldots + \beta_k X_{ki} + u_i$ specifies a linear relationship between y and $x_1, x_2, \ldots, x_k$.
2. Full Rank (No Perfect Multicollinearity)
   - All columns in $\mathbf{X}$ are linearly independent
   - $N > K$

Assumptions of the CLRM

3. $E(\mathbf{u}) = 0$

- This assumption implies that the disturbance term should have a conditional expected value of 0 at every observation.
- For the full set of observations, we can write this as:

$$E(\mathbf{u}|\mathbf{X}) = \begin{bmatrix} E[u_1|\mathbf{X}] \\ E[u_2|\mathbf{X}] \\ \vdots \\ E[u_n|\mathbf{X}] \end{bmatrix} = 0 \tag{1}$$

- The assumption in equation [1] is essential, as it implies that:

$$E(\mathbf{y}|\mathbf{X}) = \mathbf{X}\beta \tag{2}$$

Assumptions of the CLRM

4. Spherical Disturbances (Homoskedasticity and Nonautocorrelation)

- $\text{Var}(\mathbf{u}|\mathbf{X}) = \sigma^2, \text{ for all } i = 1, \ldots, n,$
- and
- $\text{Cov}(u_i, u_j|\mathbf{X}]) = 0, \text{ for all } i \neq j$
- State that the disturbance terms in the CLRM possess consistant variance and that they are uncorrelated across observations

Assumptions of the CLRM

- Additionally, these assumptions imply that:

$$E(\mathbf{uu}'|\mathbf{X}) = \begin{bmatrix} E[u_1 u_1|\mathbf{X}] & E[u_1 u_2|\mathbf{X}] & \dots & E[u_1 u_n|\mathbf{X}] \\ E[u_2 u_1|\mathbf{X}] & E[u_2 u_2|\mathbf{X}] & \dots & E[u_2 u_n|\mathbf{X}] \\ \vdots & \vdots & \vdots & \vdots \\ E[u_n u_1|\mathbf{X}] & E[u_n u_2|\mathbf{X}] & \dots & E[u_n u_n|\mathbf{X}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & 0 & \dots & 0 \\ 0 & \sigma^2 & \dots & 0 \\ & & \vdots & \\ 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

- Which we neatly summarize as:

$$E(\mathbf{uu}'|\mathbf{X}) = \sigma^2\mathbf{I} \tag{3}$$

Assumptions of the CLRM

5. Nonstochastic Regressors

- This assumption simply holds that all values in the matrix $\mathbf{X}$ are fixed
- Or: $\text{Cov}(\mathbf{X}, \mathbf{u}) = 0$
- In practice, this assumption does not match the reality of social science data where many of our independent variables of theoretical interest are random
- Thus our assumption is more about the data generating process that produces $\mathbf{x_i}$ as being fixed
- Also assumes *no measurement error*

Assumptions of the CLRM

6. Normality

- Here we simply add to the list of assumptions about the disturbances by assuming they are normally distributed
- Formally, we state:

$$\mathbf{u} \sim N[0, \sigma^2 \mathbf{I}] \quad (4)$$

OLS: [Still] Unbaised

- Start with:

$$\mathbf{Y} = \mathbf{X}\beta + \mathbf{u}$$

- Substitute OLS $\hat{\beta}$:

$$\begin{aligned}\hat{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\ &= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\end{aligned}$$

OLS: [Still] Unbaised

- and so:

$$\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}.$$

- By $\text{Cov}(\mathbf{X}, \mathbf{u}) = 0$, we have $\text{E}(\hat{\boldsymbol{\beta}}) = \boldsymbol{\beta}$.

$\hat{\beta}$ is a Consistant Estimator of β

$$\hat{\beta} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \tag{5}$$

- Since $\mathbf{Y} = \mathbf{X}\beta + \mathbf{u}$:

$$\begin{aligned} \hat{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\beta + \mathbf{u}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \\ &= \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u} \end{aligned} \tag{6}$$

$\hat{\beta}$ is a Consistant Estimator of β

- Taking expected value:

$$E[\hat{\beta} - \beta] = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'E[\mathbf{u}|\mathbf{X}] \tag{7}$$

- Since $E[\mathbf{u}|\mathbf{X}] = 0$ (by assumption):

$$E[\hat{\beta} - \beta] = 0$$
$$E[\hat{\beta}] = \beta \tag{8}$$

$\hat{\beta}$ is an Efficient Estimator of β

- In addition to Unbiasedness and Consistency, the least squares estimator is also the minimum variance, or most efficient of all unbiased linear estimators
- This can be shown via the Gauss-Markov Theorem, as we saw last week

Two Approaches

- F-test
  - Compares the model as specified (the unrestricted model) to a restricted model
  - Default in all (?) software is to effectively compare to a null model
  - This doesn't tell us much
  - Mathematically, it is pretty straightforward (we'll omit that here)

Two Approaches

- R^2
  - Often discussed as a measure of the amount of variance explained
  - Effectively calculated as $1 - \frac{\text{Residual Sum of Squared errors}}{\text{Total Sum of Squares}}$
  - Bounded by 0 (no points on regression line) and 1 (perfect fit, all points on regression line)
  - ~~Can be manipulated by~~ Increases when adding additional independent variables

Goodness-of-Fit Summarized

- All (?) software will provide an F-test, R^2 and R^2_{adj}
- Always report these
- Don't pretend that they mean more than they do.

Let's start with a toy model

```r
### Load necessary packages ----
# Use install.packages() if you do not have this package
library(tidyverse) # Data manipulation
library(stargazer) # Creates nice regression output tables
library(lmtest)    # Breusch-Pagan test
library(psych)     # Histograms and correlations for a data matrix

### Load your data ----
# We are using V-Dem version 12
my_data <- readRDS("data/vdem12.rds")

# Let's change names of some of these variables for the sake of simplicity
# I am also subsetting it to only US
us_data <- my_data |>
  filter(country_name == "United States of America") |>
  rename(democracy = v2x_polyarchy,
         gdp_per_capita = e_gdppc,
         urbanization = e_miurbani)

### Bivariate OLS ----
# Let's fit a bivariate and multivariate models
simple <- lm(democracy ~ gdp_per_capita, data = us_data)
multiple <- lm(democracy ~ gdp_per_capita + urbanization, data = us_data)

# View model summary
```

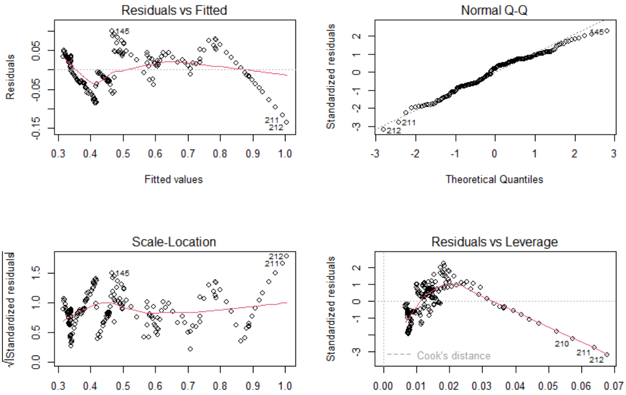
Summary of two models

Factors explaining democracy in the US

	Dependent variable:	
	Democracy	
	Simple OLS	Multiple OLS
	(1)	(2)
GDP per capita	0.012	0.013
	(0.0003)	(0.0003)
	p = 0.000*	p = 0.000*
Urbanization		0.253
		(0.056)
		p = 0.00001*
Constant	0.332	0.264
	(0.006)	(0.010)
	p = 0.000*	p = 0.000*
Observations	231	201
R2	0.904	0.942
Adjusted R2	0.904	0.942
Residual Std. Error	0.063 (df = 229)	0.044 (df = 198)
F Statistic	2,164.562* (df = 1; 229)	1,615.404* (df = 2; 198)
Notes	p < 0.05. Standard errors are in parentheses.	

Gauss-Markov assumptions with plot()

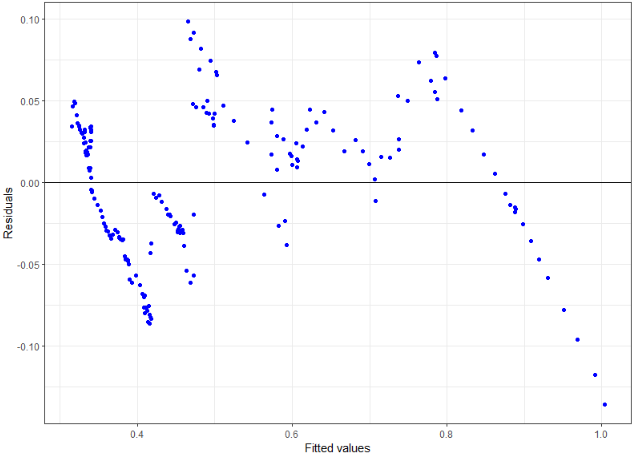
```
### Gauss-Markov assumptions using plot() ----
# Let's start with the easiest way
# plot(insert_model_name_here) function will help us to understand these assumptions
plot(multiple)

# First plot (left top corner) helps us with homoscedasticity + linearity (red line)
# Second plot (left down corner) helps us with homogeneity of variance
# Third plot (right top corner) helps us with normality of residuals
# Fourth plot (right down corner) shows outliers adn high leverage points
```



Gauss-Markov assumptions: Homoscedasticity

```r
### Gauss-Markov assumptions using other functions ----
# You can use visuals or tests
# Looking for heteroskedasticity - plotting residuals ~ fitted.values
multiple |>
  ggplot(aes(x = .fitted, y = .resid)) +
  geom_point(col = 'blue') +
  geom_abline(slope = 0) +
  labs(x = "Fitted values", y = "Residuals") +
  theme_bw()

# Perform Breusch-Pagan test
bptest(multiple)

# Since the p-value is less than 0.05, we reject the null hypothesis.
# We have sufficient evidence to say that heteroscedasticity is present in the model.
```

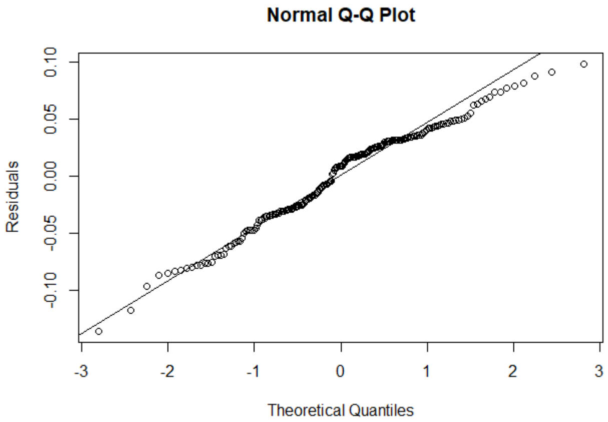
Gauss-Markov assumptions: Homoscedasticity




```
> # Perform Breusch-Pagan test
> bptest(multiple)

        studentized Breusch-Pagan test

data:  multiple
BP = 13.71, df = 2, p-value = 0.001054
```

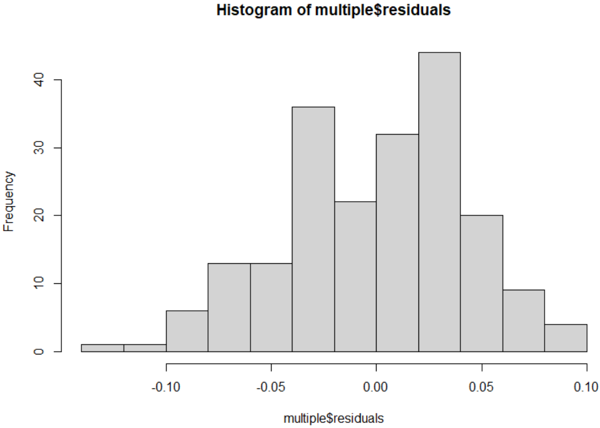
Gauss-Markov assumptions: Linearity of residuals

```
# Linearity of Relationship
# Use qqnorm and qqline to examine linearity assumption
qqnorm(residuals(multiple), ylab = "Residuals")
qqline(residuals(multiple))
```



Gauss-Markov assumptions: Normality of residuals

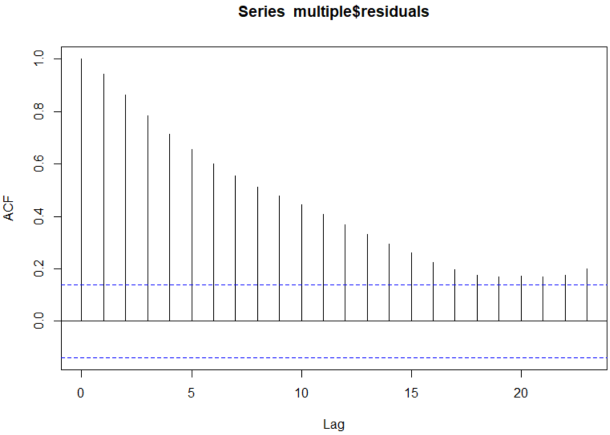
```
# Testing for multicollinearity
us_data |>
  select(democracy, gdp_per_capita, urbanization) |>
  pairs.panels(lm = T,
               method = "pearson")
```



Gauss-Markov assumptions: Autocorrelation

```r
# Serial/autocorrelation
# You can visualize or use a test
# Durbin-Watson test for autocorrelation
dwtest(multiple, data = us_data)

# Or, plot residuals for autocorrelation
stats::acf(multiple$residuals, type = "correlation")
```

```
> dwtest(multiple, data = us_data)

        Durbin-Watson test

data:  multiple
DW = 0.065127, p-value < 2.2e-16
alternative hypothesis: true autocorrelation is greater than 0
```



First, A Pretty Table

Table: A Toy Model

	Coefficient	p-Value
GDP per Capita	0.011	0.000
	(0.000)	
Urbanization	0.253	0.000
	(0.056)	
Intercept	0.264	0.000
	(0.010)	
N	201.	
R^2	0.942	
R^2_{adj}	0.942	

Note: Dependent variable is Democracy. Standard errors in parentheses.

TEX Code for Table

```
\begin{table}[h!]
    \begin{center}
        \caption{A Toy Model}
        \begin{tabular}{ l r@{.}l r@{.}l }
            \hline
            \hline
            & \multicolumn{2}{c}{Coefficient}& \multicolumn{2}{c}{$p$-Value}\\
            \hline
            GDP per Capita& 0& 011& 0 & 000\\
            &(0 & 000) &\multicolumn{2}{c}{}\\
            Urbanization& 0&    253&0 & 000 \\
            &(0 & 056) &\multicolumn{2}{c}{}        \\
            Intercept & 0&264&0&000\\
            &(0 & 010) &\multicolumn{2}{c}{}      \\
            \hline
            N & 201 & &\multicolumn{2}{c}{}\\
            $R^2$& 0 & 942&\multicolumn{2}{c}{}\\
            $R^2_{adj}$& 0 & 942&\multicolumn{2}{c}{}\\
            \hline
            \hline
        \end{tabular}\\
    \end{center}
    \medskip
    Note: Dependent variable is XXX. Standard errors in parentheses.
\end{table}
```

Interpreting Estimates

- Beyond begin BLUE, OLS is nice because of the ability to interpret coefficient estimates as independent effects
- We can write the information in the table as an equation to help think about interpretation:

$$\text{DV} = 0.264 + 0.011\text{GDP} + 0.253\text{Urban}$$

- We can thus say "a one unit increase in GDP corresponds with a 0.011 unit increase in DV."

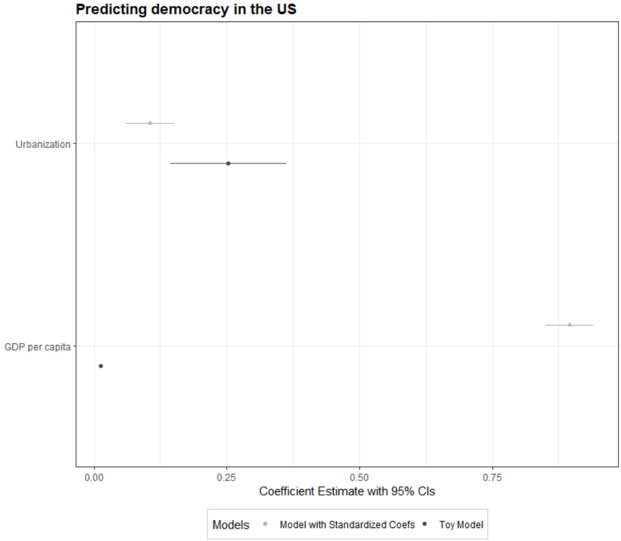
"Standardized" Coefficients

- An alternative to presenting our estimates of $\hat{\beta}$ is to present "standardized" coefficients
- The logic is to be able to compare effect sizes for things that are not on a common scale. E.g. can we say that GDP or Urbanization has a greater *substantive* effect on DV?

"Standardized" Coefficients

```r
### Standardized coefficients (beta coefs) ----

# Standardized Regression Coefficients => b_i(S_Xi/S_Y)

# There are different ways to get your beta coefficients, but use base R
streg_multiple <- lm(data.frame(scale(multiple$model))) # Get standardized regression
coefficients

# Or you could scale each variable when running lm() - like this
# lm(scale(democracy) ~ scale(gdp_per_capita) + scale(urbanization), data = us_data)

# Let's make a coefficient plot of these two models to see the difference
# I am going to use tidy() function from broom to create a nice plot
m1_tidy <- tidy(multiple) |>  mutate(model = "Toy Model")
m2_tidy <- tidy(streg_multiple) |> mutate(model = "Model with Standardized Coefs")
all_models <- bind_rows(m1_tidy, m2_tidy) # combine these models

# Let's plot
dwplot(all_models,
       show_intercept = F,
       dot_args = list(aes(colour = model, shape = model)),
       size = 3) |>
  relabel_predictors(c(urbanization = "Urbanization", gdp_per_capita = "GDP per capita")) +
  theme_bw() +
  theme(plot.title = element_text(face="bold"),
        legend.position = "bottom",
        legend.background = element_rect(colour="grey80"),
        legend.title.align = .5) +
  labs(x = "Coefficient Estimate with 95% CIs",
       y = "", title = "Predicting democracy in the US") +
  scale_shape_discrete(name  ="Models", breaks = c(0, 1)) + # breaks assign shapes
  scale_colour_grey(start = .3, end = .7, name = "Models")
```

"Standardized" Coefficients using dot-and-whisker plot



“Standardized” Coefficients

- The issue is that standardizing coefficient alters our interpretation.
- Now we can only say that “a one *standard deviation* increase in GDP the DV increases by XXXX *standard deviations*”
- Great! Now we can *sort of* directly compare effects sizes, but. . .
  - What does this mean?
  - For other issues, see King (1986, 669–674)