# Matrix Algebra

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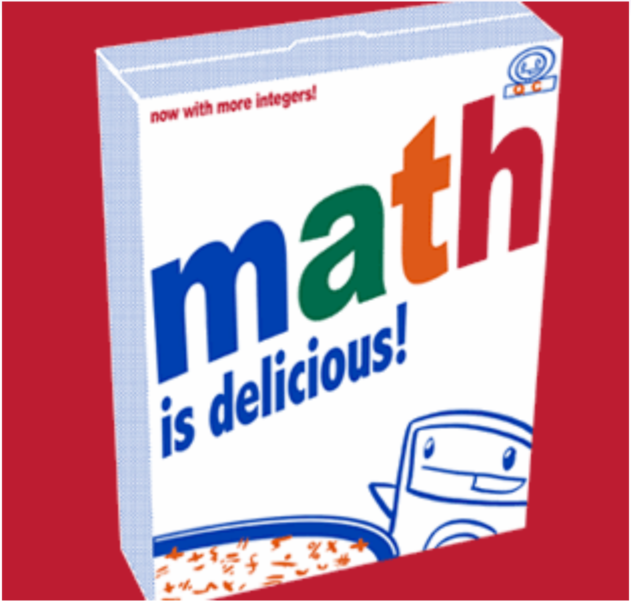
now with more integers!
math
is delicious!

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- And the same model in matrix notation:
- $\mathbf{y} = \mathbf{X}\beta + \epsilon;$
- where **y** is a vector of values for the dependent variable, **X** is a matrix of values for the independent variables, $\beta$ is vector of coefficients $\epsilon$ is a vector of disturbance terms

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- or column vectors:

$$\mathbf{v}' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

# Mathematical Operations With Vectors

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- We can add perform vector addition ($\mathbf{u} + \mathbf{v}$) or vector subtraction ($\mathbf{u} - \mathbf{v}$)
- Additionally, we can multiple or divide a vector by a scalar

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- To perform either operation the vectors *must be either both be column or both row vectors and they must have the same number of elements*
- Or, general terms, the vectors must be conformable, meaning the first vector is a size that conforms with the second
- If two vectors are nonconformable, then we cannot complete the operation

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- General Rules:

$$\mathbf{u} + \mathbf{v} = [u_1, u_2, u_3, u_4] + [v_1, v_2, v_3, v_4]$$

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$$= [-3, -1, 1, 3]$$

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$$= [3, 1.5, 1, .75]$$

# Elementary Formal Properties of Vector Algebra

| | |
|---|---|
| Commutative Property | $\mathbf{u} + \mathbf{v} = (\mathbf{v} + \mathbf{u})$ |
| Additive Associative Property | $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ |
| Vector Distributive Property | $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ |
| Scalar Distributive Property | $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$ |
| Zero Property | $\mathbf{u} + 0 = \mathbf{u} \longleftrightarrow \mathbf{u}$ - $\mathbf{u} = 0$ |
| Zero Multiplicative Property | $0\mathbf{u} = 0$ |

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- Example:

$$\underset{2 \times 2}{\mathbf{X}} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

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- Example:

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & \cdots & \cdots & \cdots & x_{1(p-1)} & x_{1p} \\ x_{21} & x_{22} & \cdots & \cdots & \cdots & x_{2(p-1)} & x_{2p} \\ \vdots & \vdots & \ddots & & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ x_{(n-1)1} & x_{(n-1)2} & \cdots & \cdots & \cdots & x_{(n-1)(p-1)} & x_{(n-1)p} \\ x_{n1} & x_{n2} & \cdots & \cdots & \cdots & x_{n(p-1)} & x_{np} \end{bmatrix}$$

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$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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# Properties of (Conformable) Matrix Manipulation

| | |
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| Commutative Property | $\mathbf{X} + \mathbf{Y} = (\mathbf{Y} + \mathbf{X})$ |
| Additive Associative Property | $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\ \mathbf{Y} + \mathbf{Z})$ |
| Matrix Distributive Property | $s(\mathbf{X} + \mathbf{Y}) = s\mathbf{X} + s\mathbf{Y}$ |
| Scalar Distributive Property | $(s + t)\mathbf{X} = s\mathbf{X} + t\mathbf{X}$ |
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- This generates the second issue: the order matters. $\mathbf{XY} \neq \mathbf{YX}$
- For example: $\underset{(k \times n)}{\mathbf{X}} \underset{(n \times p)}{\mathbf{Y}}$ would be valid, but $\underset{(n \times p)}{\mathbf{Y}} \underset{(k \times n)}{\mathbf{X}}$ would not be (unless $k = p$)

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- Formally, if we define a matrix **Z** as the product of **XY**, then
  - We can define each element of **Z** as $\mathbf{z}_{kp} = \sum_n x_{kn} y_{np}$
  - In words: The element in the *k*th row and the *p*th column of **Z** is obtained by multiplying the elements of the *k*th row of **X** by the corresponding elements of the *p*th column of **Y** and summing over all terms

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- In general:

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$$= \begin{bmatrix} -2 & 4 \\ -6 & 10 \end{bmatrix}$$

# Properties of (Conformable) Matrix Multiplication

| | |
|---|---|
| Associative Property | $(\mathbf{XY})\mathbf{Z} = \mathbf{X}(\mathbf{YZ})$ |
| Additive Distributive Property | $(\mathbf{X} + \mathbf{Y})\mathbf{Z} = \mathbf{XZ} + \mathbf{YZ}$ |
| Scalar Distributive Property | $s\mathbf{XY} = (\mathbf{X}s)\mathbf{Y} = \mathbf{X}(s\mathbf{Y}) = \mathbf{XY}s$ |
| Zero Multiplicative Property | $\mathbf{X}0 = 0$ |

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- Example:

$$\mathbf{X} = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 1 & 3 \end{bmatrix} \qquad \mathbf{X}' = \begin{bmatrix} 6 & 2 \\ 1 & 3 \\ 3 & 5 \end{bmatrix}$$

# Properties of Matrix Transposition

| | |
|---|---|
| Invertability | $(\mathbf{X}')' = \mathbf{X}$ |
| Additive Property | $(\mathbf{X} + \mathbf{Y})' = \mathbf{X}' + \mathbf{Y}'$ |
| Multiplicative Property | $(\mathbf{XY})' = \mathbf{Y}'\mathbf{X}'$ |
| General Multiplicative Property | $(\mathbf{X}_1\mathbf{X}_2 \dots \mathbf{X}_{n-1}\mathbf{X}_n)' = \mathbf{X}_n'\mathbf{X}_{n-1}' \dots \mathbf{X}_2'\mathbf{X}_1'$ |

# Matrix Inversion

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- Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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- Moving to larger matrices, the process of calculating a determinant become significantly more complex
- And involves the defining of submatrices created by deleting specific rows and columns.
- Thankfully, modern technology has largely elemented the need to hand-calculate determinants.